

# A convolution integral representation of the thermal Sunyaev-Zel'dovich effect

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## Abstract

Analytical expressions for the non-relativistic and relativistic Sunyaev-Zel'dovich effect (SZE) are derived by means of suitable convolution integrals. The establishment of these expressions is based on the fact that the SZE disturbed spectrum, at high frequencies, possesses the form of a Laplace transform of the single line distortion profile (structure factor). Implications of this description of the SZE related to light scattering in optically thin plasmas are discussed.

## 1 Introduction

Distortions to the cosmic microwave background radiation (CMBR) spectrum arise from the interaction between the radiation photons and electrons present in large structures such as the hot intracluster gas now known to exist in the universe. Such distortions are only a very small effect changing the brightness of the spectrum by a figure of the order of 0.1 percent. This effect, excluding the proper motion of the cluster, is now called the thermal Sunyaev-Zel'dovich effect [1] [2] and its detection is at present a relatively feasible task due to the modern observational techniques available. Its main interest lies on the fact that it provides information to determine important cosmological parameters such as Hubble's constant and the baryonic density [4]-[6].

The kinetic equation first used to describe the SZE was one derived by Kompaneets back in 1956 [3] [7]-[8]. This approach implies a photon diffusion

description of the effect that works basically when the electrons present in the hot intracluster gas are non-relativistic. Many authors, including Sunyaev and Zel'dovich themselves, were very reluctant in accepting a diffusive mechanism as the underlying phenomena responsible for the spectrum distortion. Later on, Rephaeli and others [9] computed the distorted spectrum by considering Compton scattering off relativistic electrons. Those works show that, at high electron temperatures, the distortion curves are significantly modified. Useful analytic expressions that describe the relativistic SZE can be easily found in the literature [10].

One possible physical picture of the effect is that of an absorption-emission process in which a few photons happen to be captured by electrons in the optically thin gas. An electron moving with a given thermal velocity emits (scatters) a photon with a certain incoming frequency  $\nu_o$  and outgoing frequency  $\nu$ . The line breath of this process is readily calculated from kinetic theory taking into account that the media in which the process takes place has a small optical depth directly related to the Compton parameter  $y$ . When the resulting expression, which will be called *structure factor*, is convoluted with the incoming flux of photons obtained from Planck's distribution, one easily obtains the disturbed spectra. For the non-relativistic SZE, a Gaussian structure factor has been successfully established [11], but the obtention of a simple analytic relativistic structure factor is a more complicated task [12]. Nevertheless, this work shows that a relativistic structure factor can be obtained with the help of the expressions derived in Refs. [10] and by the use of simple mathematical properties of the convolution integrals that describe the physical processes mentioned above. In both cases, extensive use has been made of the intracluster gas.

To present these ideas we have divided the paper as follows. In section II, for reasons of clarity we summarize the ideas leading to the SZ spectrum for a non-relativistic electron gas emphasizing the concept of structure factor. In section III we discuss the relativistic case using an appropriate equation for the structure factor whose full derivation is still pending but leads to results obtained by other authors using much more complicated methods. Section IV is left for concluding remarks.

## 2 General Background

As it was clearly emphasized by the authors of this discovery in their early publications [1] [2] as well as by other authors, the distortion in the CMBR spectrum by the interaction of the photons with the electrons in the hot plasma filling the intergalactic space is due to the diffusion of the photons in the plasma which, when colliding with the isotropic distribution of a non-relativistic electron gas, generates a random walk. The kinetic equation used to describe this process was one first derived by Kompaneets back in 1956 [7]-[8]. For the particular case of interest here, when the electron temperature  $Te \sim 10^8 K$  is much larger

than that of the radiation,  $T \equiv T_{Rad}$  (2.726 K), the kinetic equation reads [3]

$$\frac{dN}{dy} = \nu^2 \frac{d^2 N}{d\nu^2} + 4\nu \frac{dN}{d\nu} \quad (1)$$

Here  $N$  is the Bose factor,  $N = (e^x - 1)^{-1}$ ,  $x = \frac{h\nu}{kT}$ ,  $\nu$  is the frequency,  $h$  is Planck's constant,  $k$  Boltzmann's constant and  $y$  the "Compton parameter" given by,

$$y = \frac{k T_e}{m_e c^2} \int \sigma_T n_e c dt = \frac{k T_e}{m_e c^2} \tau \quad (2)$$

$m_e$  being the electron mass,  $c$  the velocity of light,  $\sigma_T$  the Thomson's scattering cross section and  $n_e$  the electron number density. The integral in Eq. (2) is usually referred to as the "optical depth,  $\tau$ ", measuring essentially how far in the plasma can a photon travel before being captured (scattered) by an electron. Without stressing the important consequences of Eq. (1) readily available in many books and articles [3]-[5], we only want to state here, for the sake of future comparison, that since the observed value of  $N_o(\nu)$  is almost the same as its equilibrium value  $N_{eq}(\nu)$ , to a first approximation one can easily show that

$$\frac{\delta N}{N} \equiv \frac{N(\nu) - N_{eq}(\nu)}{N_{eq}(\nu)} = y \left[ \frac{x^2 e^x (e^x + 1)}{(e^x - 1)^2} - \frac{4x e^x}{e^x - 1} \right] \quad (3)$$

implying that, in the Rayleigh-Jeans region ( $x \ll 1$ ),

$$\frac{\delta N}{N} \cong -2y \quad (4)$$

and in the Wien limit ( $x \gg 1$ ) :

$$\frac{\delta N}{N} \cong x^2 y \quad (5)$$

If we now call  $I_o(\nu)$  the corresponding radiation flux for frequency  $\nu$ , defined as

$$I_o(\nu) = \frac{c}{4\pi} U_v(T) \quad (6)$$

where  $U_v(T) = \frac{8\pi h\nu^3}{c^3} N_{eq}(\nu)$  is the energy density for frequency  $\nu$  and temperature  $T$ , and noticing that

$$\frac{\delta T}{T} = \left( \frac{\partial(\ln I(\nu))}{\partial(\ln T)} \right) \left( \frac{\delta I}{I} \right) \quad (7)$$

we have that the change in the background brightness temperature is given, in the two limits, by

$$\frac{\delta T}{T} \cong -2y, \quad x \ll 1 \quad (8)$$

and

$$\frac{\delta T}{T} \cong x y, \quad x \gg 1 \quad (9)$$

showing a decrease in the low frequency limit and an increase in the high frequency one. Finally, we remind the reader that the curves extracted from Eq. (3) for reasonable values of the parameter “ $y$ ” show a very good agreement with the observational data.

As mentioned before, many authors, including Sunyaev and Zel’dovich themselves, were very reluctant in accepting a diffusive mechanism as the underlying phenomena responsible for the spectrum distortion. The difficulties of using diffusion mechanisms to study the migration of photons in turbid media, specially thin media, have been thoroughly underlined in the literature [13]-[15]. Several alternatives were discussed in a review article in 1980 [5] and a model was set forth by Sunyaev in the same year [16] based on the idea that Compton scattering between photons and electrons induce a change in their frequency through the Doppler effect. Why this line of thought has not been pursued, or at least, not widely recognized, is hard to understand. Two years ago, one of us (ASV) [11] reconsidered the single scattering approach to study the SZ effect. The central idea in that paper is that in a dilute gas, the scattering law is given by what in statistical physics is known as the *dynamic structure factor*, denoted by  $G(k, \nu)$  where  $k = \frac{2\pi}{\lambda}$ , and  $\lambda$  is the wavelength. In such a system, this turns out to be proportional to  $\exp(-\frac{(\nu-\bar{\nu})^2}{W^2})$  where  $W$  is the broadening of the spectral line centered at frequency  $\nu$  given by

$$W = \frac{2}{c} \left( \frac{2kT_e}{m} \right)^{1/2} \nu \quad (10)$$

If one computes the distorted spectrum through the convolution integral

$$I(\nu) = \int_0^\infty I_o(\bar{\nu}) G(k, \bar{\nu} - (1 - ay)\nu) d\bar{\nu} \quad (11)$$

where the corresponding frequency shift  $\frac{\delta\nu}{\nu}$  has been introduced through Eqs. (8-9) ( $a = -2 + x$ ), one gets a good agreement with the observational data. This result is interesting from, at least, two facts. One, that such a simple procedure is in agreement with the diffusive picture. This poses interesting mathematical questions which will be analyzed elsewhere, specially since the exact solution to Eq. (1) is known (see Eq. (A-8) ref. [5]). The other one arises from the fact that this is what triggered the idea of reanalyzing the SZ effect using elementary arguments of statistical mechanics and constitutes the core of this paper.

Indeed, we remind the reader that if an atom in an ideal gas moving say with speed  $u_x$  in the  $x$  direction emits light of frequency  $\nu_o$  at some initial speed  $u_x(0)$ , the intensity of the spectral line  $I(\nu)$  is given by

$$\frac{I(\nu)}{I_o(\nu)} = \exp \left[ -\frac{m c^2}{2kT} \left( \frac{\nu_o - \nu}{\nu} \right)^2 \right] \quad (12)$$

Eq. (12) follows directly from the fact the velocity distribution function in an ideal gas is Maxwellian and that  $u_x$ ,  $u_x(0)$  and  $\nu$  are related through the Doppler effect. For the case of a beam of photons of intensity  $I_o(\nu)$  incident

on a hot electron gas regarded as an ideal gas in equilibrium at a temperature  $T_e$  the full distorted spectrum may be computed from the convolution integral given by

$$I(\nu) = \frac{1}{\sqrt{\pi}W(\nu)} \int_0^\infty I_o(\bar{\nu}) \exp \left[ - \left( \frac{\bar{\nu} - f(y)\nu}{W(\nu)} \right)^2 \right] d\bar{\nu} \quad (13)$$

which defines the joint probability of finding an electron scattering a photon with incoming frequency  $\bar{\nu}$  and outgoing frequency  $\nu$  multiplied by the total number of incoming photons with frequency  $\bar{\nu}$ .  $\frac{1}{\sqrt{\pi}}W(\nu)$  is the normalizing factor of the Gaussian function for  $f(y) = 1$ ,  $W(\nu)$  is the width of the spectral line at frequency  $\nu$  and its squared value follows from Eq.(10)

$$W^2(\nu) = \frac{4kT_e}{m_e c^2} \tau \nu^2 = 4y \nu^2 \quad (14)$$

where  $\tau$  is the optical depth whose presence in Eq. (14) will be discussed later. The function  $f(y)$  multiplying  $\nu$  in Eq. (13) is given by  $f(y) = 1 + ay$  where  $a = -2$  in the Rayleigh-Jeans limit and  $a = xy$  in the Wien's limit, according to Eqs. (8-9) and the fact that  $\frac{\Delta\nu}{\nu} = \frac{\Delta T}{T}$  for photons. Eq.(13) is the central object of this paper so it deserves a rather detailed examination. In the first place it is worth noticing that  $I_o(\nu)$ , the incoming flux, is defined in Eq. (6). Secondly, it is important to examine the behavior of the full distorted spectrum in both the short and high frequency limits. In the low frequency limit, the Rayleigh-Jeans limit  $I_o(\nu) = \frac{2kT\nu^2}{c^2}$ , so that performing the integration with  $a = -2$  and noticing that  $y$  is a very small number, one arrives at the result

$$\frac{\delta I}{I} \equiv \frac{I(\nu) - I_o(\nu)}{I_o(\nu)} = -2y, \quad x \ll 1 \quad (15)$$

(15) is in complete agreement with the value obtained using Eq.(3), the photon diffusion equation. In the high frequency limit where  $a = xy$  and  $I_o(\nu) = \frac{2h\nu^3}{c^2} e^{-x}$ , a slightly more tedious sequence of integrations leads also to a result at grips with the diffusion equation, namely

$$\frac{\delta I}{I} = x^2 y, \quad x \gg 1 \quad (16)$$

Why both asymptotic results, the ones obtained with the diffusion equation and those obtained from Eq. (13) agree so well, still puzzles us. At this moment we will simply think of them as a mathematical coincidence. Nevertheless, it should be stressed that in his 1980 paper, Sunyaev [16] reached rather similar conclusions although with a much more sophisticated method, and less numerical accuracy for the full distortion curves. The distorted spectrum for the CMBR radiation may be easily obtained by numerical integration of Eq. (13) once the optical depth is fixed,  $y$  is determined through Eq.(2) and  $a = -2 + x$ . The intergalactic gas cloud in clusters of galaxies has an optical depth  $\tau \sim 10^{-2}$  [9].

From the results thus obtained in the non-relativistic case, one appreciates the rather encouraging agreement between the observational data and the theoretical results obtained with the three methods, the diffusion equation, the structure factor or scattering law approach, and the Doppler effect. This, in our opinion is rather rewarding and some efforts are in progress to prove the mathematical equivalence of the three approaches. From the physical point of view, and for reasons already given by many authors, we believe that the scattering Doppler effect picture does correspond more with reality, specially for reasons that will become clear in the last section.

### 3 The relativistic case

Following the discussion of the previous section, Eq.(13) may be written as:

$$I(\nu) = \int_0^\infty I_o(\bar{\nu}) G(\bar{\nu}, \nu) d\bar{\nu} \quad (17)$$

where the function  $G(\bar{\nu}, \nu) = \frac{1}{\sqrt{\pi W(\nu)}} \exp \left[ - \left( \frac{\bar{\nu} - f(y)\nu}{W(\nu)} \right)^2 \right]$  and  $I_o(\bar{\nu})$  is defined in Eq.(6). It is now clear that in Wien's limit, when  $\frac{h\nu}{kT} \gg 1$ , Eq. (17) becomes the Laplace transform of  $I_o(\bar{\nu})$  with a parameter  $s = \frac{h}{kT}$  so that calling  $I_{pW}$  the distorted spectrum in that limit, one gets that:

$$I_{pW}(\nu) = \frac{2h}{c^2} \int_0^\infty \bar{\nu}^3 G(\bar{\nu}, \nu) e^{-s\bar{\nu}} d\bar{\nu} \quad (18)$$

Comparing Eq. (18) with the result obtained by integrating the Kompaneets equation for the change of intensity in Wien's limit, namely [2]-[3]

$$\Delta I = y \frac{2k^3 T_o^3}{h^2 c^2} x^4 \frac{e^x}{(e^x - 1)^2} [-4 + F(x)] \quad (19)$$

where  $F(x) = x \coth(\frac{x}{2})$ , a straight forward inversion of the Laplace transform yields that

$$G(\bar{\nu}, \nu) = \delta(\bar{\nu} - \nu) - 4y \frac{\nu^4}{\bar{\nu}^3} \delta'(\bar{\nu} - \nu) + y \frac{\nu^5}{\bar{\nu}^3} \delta''(\bar{\nu} - \nu) \quad (20)$$

In this equation, the primes denote derivatives with respect to  $\nu$ . It may be thus regarded as a representation of  $G(\bar{\nu}, \nu)$  in terms of the Delta function and its derivatives. To illustrate this point we can show that the RJ and Wien's limits of integral (11) arise by substituting  $G(\bar{\nu}, \nu)$  by a shifted delta function (see appendix).

In the relativistic case, the form for  $G(\bar{\nu}, \nu)$  may also be written down taking the relativistic form for the particle's energy  $E = \frac{m_o c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$  and performing a

power series expansion in powers of  $\frac{y^2}{c^2}$ . Defining  $z = \frac{kT_e}{m_e c^2}$ , this leads to an integral for  $I(\nu)$  which reads as follows:

$$I(\nu) = \frac{1}{\sqrt{\pi}W(\nu)} \int_0^\infty I_o(\bar{\nu}) \exp \left[ -\frac{\sqrt{1 - \frac{[\bar{\nu} - f(y)\nu]^2}{2\tau y}} - 1}{z} \right] d\bar{\nu} \quad (21)$$

Nevertheless, its integration even in the two limits Wien's and Rayleigh-Jeans has been unsuccessful. However, by direct numerical integration one obtains curves that agree qualitatively well with those obtained by other methods. Thus, in order to verify the generosity of the delta function representation we proceeded in an indirect way. It has been shown in the literature that a good representation of the relativistic SZE up to second order in  $y$  reads as [10]:

$$\Delta I = y \frac{2k^3 T_e^3}{h^2 c^2} x^4 \frac{e^x}{(e^x - 1)^2} [-4 + F(x) + \frac{y^2}{\tau} (-10 + \frac{47}{2} F(x) - \frac{47}{5} F^2(x) + \frac{7}{10} F^3(x) - \frac{21}{5} H^2(x) + \frac{7}{5} F(x) H^2(x))] \quad (22)$$

Here,  $H(x) = x [\sinh(x/2)]^{-1}$ . Taking the inverse Laplace transform in Wien's limit of Eq. (22) one obtains that

$$G_R(\bar{\nu}, \nu) = \delta(\bar{\nu} - \nu) - \left( 4y + \frac{10y^2}{\tau} \right) \frac{\nu^4}{\bar{\nu}^3} \delta'(\bar{\nu} - \nu) + \left( y + \frac{47y^2}{2\tau} \right) \frac{\nu^5}{\bar{\nu}^3} \delta''(\bar{\nu} - \nu) - \frac{42y^2}{5\tau} \frac{\nu^6}{\bar{\nu}^3} \delta'''(\bar{\nu} - \nu) + \frac{7y^2}{10\tau} \frac{\nu^7}{\bar{\nu}^3} \delta^{(IV)}(\bar{\nu} - \nu) \quad (23)$$

There is of course the remaining problem, namely to obtain Eq. (22) from the convolution integral (21). This has so far defied our skills, but is purely a mathematical problem. The main point we want to underline here is the possibility of using expressions such as Eqs. (20) and (23), in principle obtainable from convolution integrals with structure factors which provide a useful analytical tool that avoids resorting too complicated and laborious methods. In fact, Fig. 1 shows CMBR distortion comparison curves obtained using Eqs. (19) and (17) with (20), for several non-relativistic clusters. Fig. 2 shows the corresponding comparison relativistic curves at higher, realistic, electron temperatures using Eqs. (22) and (17) with (23). It is clear that, for any practical purpose, the curves are identical. Indeed, one is not able to identify two different plots in each figure.

## 4 Discussion of the results

It is known that, for the case of a beam of photons of intensity  $I_o(\nu)$  incident on a hot electron gas regarded as an ideal gas in equilibrium at a temperature  $T_e$ , the full non-relativistic SZE distorted spectrum may be computed from the convolution integral given in Eq. (13), which defines the joint probability of

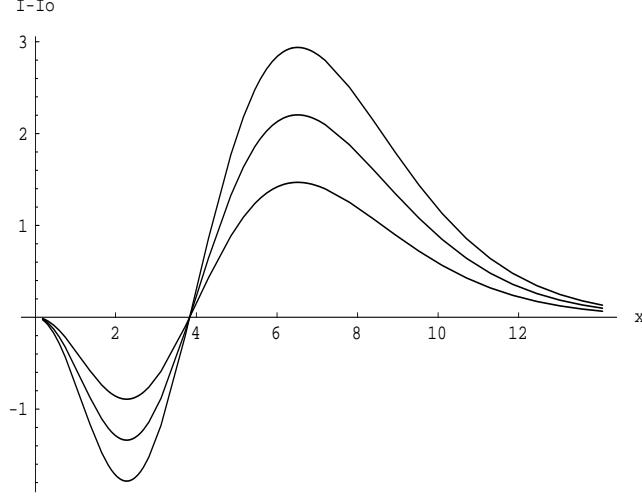


Figure 1: CMBR SZE distortion as computed by Eqs. (5,10,12), with  $\tau = 10^{-2}$ ,  $T = 4KeV$  (lower curve),  $T = 6KeV$  (middle curve) and  $T = 8KeV$  (upper curve). No difference exists if compared the corresponding plot of Eq. (19).  $\delta I$  is measured in  $erg\ s^{-1}\ cm^{-2}\ ster^{-1}$ . The figure is scaled by a factor of  $10^{18}$ .

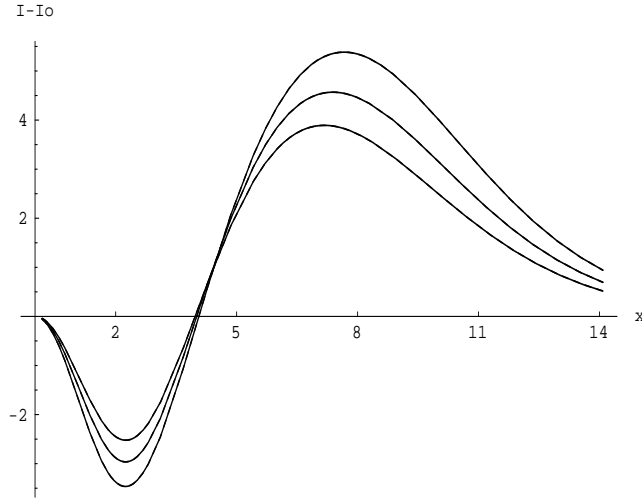


Figure 2: Relativistic CMBR SZE distortion as computed by Eqs. (5,11,13), with  $\tau = 10^{-2}$ ,  $T = 12.5KeV$  (lower curve),  $T = 15KeV$  (middle curve) and  $T = 18KeV$  (upper curve). No difference exists if compared the corresponding plot of Eq. (22).  $\delta I$  is measured in  $erg\ s^{-1}\ cm^{-2}\ ster^{-1}$ . The figure is again scaled by a factor of  $10^{18}$ .



finding an electron scattering a photon with incoming frequency  $\bar{\nu}$  and outgoing frequency  $\nu$  multiplied by the total number of incoming photons with frequency  $\bar{\nu}$ .  $1 - 2y$  is a function that corrects the outgoing photon frequency due to the thermal effect [5].  $\frac{1}{\sqrt{\pi}}W(\nu)$  is the normalizing factor of the Gaussian,  $W(\nu)$  is the width of the spectral line at frequency  $\nu$  and its squared value is defined in Eq. (14).

Thus, it is clear that an accurate convolution integral between the undistorted Planckian and a *regular function* exists and can be obtained from strictly physical arguments in the non-relativistic SZE. It is interesting, however, that the corresponding relativistic expression does not seem to be a simple physical generalization of Eq. (13). Yet, Eqs. (17,23) give the correct mathematical description of the relativistic distortion by means of Dirac delta functions and its derivatives. Two questions can now be posed. One, of a rather mathematical fashion, states under what conditions a structure factor such as that appearing in Eq. (17) can be written in a representation involving Dirac's delta functions and its derivatives. This subject seems to be related to the mathematical theory of distributions. The second one, of more physical type, concerns with the possible description of the thermal SZE as a light scattering problem in which CMBR photons interact with an electron gas in a thermodynamical limit that allows simple expressions for the structure factors. If this were the case, the thermal relativistic SZE physics would not need of Montecarlo simulations or semi-analytic methods in its convolution integral description.

Emphasis should be made on the fact that the main objective pursued in this work is to enhance the physical aspects of the SZE by avoiding complicated numerical techniques in many cases. This is achieved, as shown, by resorting the concept of structure factors, or scattering laws, widely used in statistical physics.

## APPENDIX

The starting point here is Eq.(17). In the RJ limit,  $I_{oRJ}(\bar{\nu}) = \frac{2kT}{c^2}\bar{\nu}^2$ . Introducing  $G(\bar{\nu}, \nu) = \delta(\bar{\nu} - \nu(1 - ay))$  one obtains

$$I_{RJ}(\nu) = \frac{2kT}{c^2} \int_0^\infty \bar{\nu}^2 \delta(\bar{\nu} - \nu(1 - ay)) d\bar{\nu} = \frac{2kT}{c^2} \nu^2 (1 - ay)^2$$

keeping terms linear in  $y$  we find that

$$\frac{I_{RJ}(\nu) - I_{oRJ}(\nu)}{I_{oRJ}(\nu)} = -2ay$$

This result is consistent with Eq. (8) setting  $a = 1$ .

Now, in the Wien limit,  $I_{oW}(\bar{\nu}) = \frac{2h}{c^2}\bar{\nu}^3 e^{-\frac{h\bar{\nu}}{kT}}$ , in this case the introduction of  $\delta(\bar{\nu} - \nu(1 - ay))$  leads to

$$I_W(\nu) = \frac{2h}{c^2} \int_0^\infty \bar{\nu}^3 e^{-\frac{h\bar{\nu}}{kT}} \delta(\bar{\nu} - \nu(1 - ay)) d\bar{\nu} = \frac{2h}{c^2} \nu^3 (1 - ay)^3 e^{-\frac{h\nu}{kT}} e^{\frac{h\nu}{kT} ay}$$

Expanding  $e^{-\frac{h}{kT}ay}$  and keeping terms up to first order in  $y$  we obtain that

$$\frac{I_W(\nu) - I_{oW}(\nu)}{I_{oW}(\nu)} = -6ay + axy$$

Finally, setting  $a = 1$  and considering  $x \gg 6$  in the Wien limit, we obtain the desired result, Eq. (9).

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## References

- [1] Y.B. Zel'dovich and R.A. Sunyaev, *Astrophys. Space Sci.* **4**, 301 (1969).
- [2] R.A. Sunyaev and Ya. B. Zel'dovich. *Comm. Astrophys. Space Phys.* **4**, 173 (1972).
- [3] P.J.E. Peebles, *Physical Cosmology* (Princeton Univ. Press, Princeton, 2nd. Ed. 1993).
- [4] M.M. Joy and J.E. Carlstrom; *Science* **291**, 1715 (2001) and ref. therein.
- [5] R.A. Sunyaev and Ya. B. Zel'dovich; *Ann. Rev. Astron. Astrophys.* **18**, 537 (1980)
- [6] S.H. Hansen, S. Pastor and D.V. Semikoz, *Astrophys. J.* **573**, (2002) L69 [astro-ph/0205295]
- [7] A.S. Kompaneets; *JETP.* **4**, 730 (1957).
- [8] R. Weymann; *Phys. Fluids* **8**, 2112 (1965)
- [9] Y. Rephaeli, *Astrophys. J.* **445**, 33 (1995);  
A.D. Challinor and A.N. Lasenby, *Astrophys. J.* **499**, 1 (1998);  
N. Itoh, Y. Kohyama, and S. Nozawa, *Astrophys. J.* **502**, 7 (1998);  
S. Nozawa, N. Itoh, and Y. Kohyama, *Astrophys. J.* **507**, 530 (1998);  
G. Rybicky, (2002) astro-ph/0208542  
A.D. Dolgov, S.H. Hansen, D.V. Semikoz and S. Pastor, *Astrophys. J.* **554** 74 (2001) [astro-ph/0010412].
- [10] S.Sazonov and R.A Sunyaev (1998) astro-ph/9804125 ;  
E.S. Battistelli, M. DePetrìs, L. Lamagna, F. Melchiorri, E. Palladino, G. Savini, A. Cooray, A.Melchiorri, Y. Rephaeli, M. Shimon (2002) astro-ph/0208027 and references therein. Submitted to ApJL.
- [11] A. Sandoval-Villalazo *Physica A* **313** 456-462 (2002)  
A. Sandoval-Villalazo and L.S. García-Colín (2002) [astro-ph/0207218].
- [12] S.M. Molnar and M. Birkinshaw (1999) [astro-ph/9903444];
- [13] G. H. Weiss; *Physica A* **311**, 381 (2002).

- [14] D.J. Durian and J. Durnick, J. Opt. Soc. Am. **A14** 235 (1997).
- [15] A.H. Gandibache and G.H. Weiss in Progress in Optics XXXIV; E. Wolf, Ed. (Elsevier, Amsterdam 1995) p. 332.
- [16] R.A. Sunyaev; Sov. Astron. Lett. **6**, 212 (1980).